A Short Note on Measure Theory

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Abstract

Here is my midterm review sheet for MIT's *Measure Theory and Integration (18.125)*, Spring 2023. This may contain typos and occasional mistakes, as it was written during my midterm prep. This note only covers the course's first half, not the applications (Functional Analysis, Fourier analysis, etc.). So, don't study just this note for the final preparations, as you will be woefully under-prepared. Feel free to copy and distribute under the MIT license, but I would appreciate it if you give me credit! Please report typos and errors to `notadib@mit.edu`.

1 Introduction

1.1 Sigma Algebra

Let E be a set. A σ-algebra on E, called $\mathcal{E}$, is a subset of $P(E)$ with the following properties:

- $\varnothing \in \mathcal{E}$
- $\forall A \in \mathcal{E}$, we have $E \backslash A \in \mathcal{E}$.
- For distinct $(A_n)_{n\in\mathbb{N}} \in \mathcal{E}$, we have $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

$(E, \mathcal{E})$ **is called a measurable space and subsets of** $\mathcal{E}$ **are called measurable sets.**

Properties

- For $(A_n)_{n\in\mathbb{N}} \in \mathcal{E}$, we have $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{E}$ (follows from the 3rd property above).
- If $A, B \in \mathcal{E}$, then we have $A \cap B \in \mathcal{E}$ and $A \cup B \in \mathcal{E}$.
- $(\mathcal{E}_i)_{i\in I}$ is a countable family of σ-algebras, then $\bigcap_{i\in I} \mathcal{E}_i$ is a σ-algebra (check three properties above).

Remarks

- Let $\mathcal{C} \in P(E)$. The σ-algebra generated by $\mathcal{C}$, called $\sigma(\mathcal{C})$, is the following:

$$\sigma(\mathcal{C}) = \bigcap \{\mathcal{E} | \mathcal{E} \text{ is } \sigma\text{-algebra on } E \text{ and } \mathcal{C} \in \mathcal{E}\}$$

- To prove every $A \in \sigma(\mathcal{C})$ satisfies some property $p(A)$, first show that $\forall A \subseteq \mathcal{C}$, $p(A)$ holds and $\{A \in \sigma(\mathcal{C}) | p(A) \text{ holds}\}$ is a σ-algebra.

1.2 Measure, Borel Measure, Lebesgue Measure, Probability Measure

Measure: Let $(E, \mathcal{E})$ be a measurable space. A measure μ on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ such that

- $\mu(\varnothing) = 0$
- If $(A_n)_{n \geq 0}$ is a countable family of distinct elements of E, then

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n)$$

Furthermore $(E, \mathcal{E}, \mu)$ is called a measured space.

Properties:

1. If $A, B \in \mathcal{E}$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$.
2. If $A, B \in \mathcal{E}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
3. If $(A_n)_{n \geq 0}$ is an increasing sequence of measurable sets, that is $A_n \subseteq A_{n+1}$, then

$$\mu\left(\bigcup_{n \geq 0} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

4. If $(A_n)_{n \geq 0}$ is a decreasing sequence of measurable sets, that is $A_{n+1} \subseteq A_n$, and $\mu(A_0) < \infty$ then

$$\mu\left(\bigcap_{n \geq 0} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

5. (Borel-Cantelli Lemma) If $(A_n)_{n \geq 0}$ is a sequence of measurable sets such that $\sum_{n=0}^{\infty} \mu(A_n) < \infty$, then

$$\mu\left(\bigcap_{n \geq 0} \bigcup_{k \geq n} A_k\right) = 0$$

Borel Measure: Let $\mathcal{E} = \{(-\infty, a]; a \in \mathbb{R}\}$. It's easy to show that $\sigma(\mathcal{E})$ contains all open intervals of $\mathbb{R}$ and is called the Borel σ-algebra of $\mathbb{R}$. It is written as $\mathcal{B}(\mathbb{R})$.

In general, $\mathcal{B}(X) = \sigma(T)$ where $T = \{\text{Open sets of } X\}$. Any measure μ on $(X, \mathcal{B}(X))$ is called a Borel measure on X.

Regularity of Borel Measure: Let (X, d) be a metric space and μ be a finite Borel measure on (X, d). Then, for every $A \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is a closed set F and an open set U such that $F \subseteq A \subseteq U$ and $\mu(U \backslash F) \leq \varepsilon$. In other words, we can approximate Borel measurable sets arbitrarily closely with an open set and a closed set.

For proof consider $\mathcal{E}$ to be the set of sets that satisfy the given properties and then show $\mathcal{E}$ is a sigma-algebra and it contains all the open sets of $\mathcal{B}(X)$.

Lebesgue Measure: Let μ be a Borel measure on $\mathbb{R}$. Then the following properties are equivalent:

1. For every $a < b$, $\mu((a, b]) = b - a$
2. $\mu([0, 1]) = 1$ and μ is invariant by translation.

There is a unique Borel measure on $\mathbb{R}$ satisfying 1 and 2. This unique measure is called the Lebesgue measure on $\mathbb{R}$ and is written as λ. Note that showing the existence of λ requires Axiom of Choice.

Probability Measure: A measure μ on a measurable space $(E, \mathcal{E})$ such that $\mu(E) = 1$. Note that due to the monotonicity of measure, this means for any $X \in \mathcal{E}$, we have $0 \leq \mu(X) \leq 1$, and $\mu(X)$ is called the probability of event X.

1.3 Pi System and Lambda System

Let E be a set and $\mathcal{E} \subseteq P(E)$. We say that $\mathcal{E}$ is a Π-*system* if

- $E \in \mathcal{E}$
- $\forall A, B \in \mathcal{E}$, we have $A \cap B \in \mathcal{E}$.

We say that $\mathcal{E}$ is a Λ-*system* if

- $E \in \mathcal{E}$
- $\forall A, B \in \mathcal{E}$ and $A \cap B = \varnothing$, we have $A \cup B \in \mathcal{E}$.
- If $A, B \in \mathcal{E}$ and $A \subseteq B$, then $B \backslash A \in \mathcal{E}$.
- If $(A_n)_{n \geq 0}$ is an increasing sequence of elements of $\mathcal{E}$, then $\bigcup_{n \geq 0} A_n \in \mathcal{E}$.

We need these constructions because it is sometimes easier to prove that a family of sets is either a Π-system or a Λ-system than σ-algebra. Then we have the following important results:

Lemma 1. *Let E be a set and $\mathcal{E} \subseteq P(E)$. If $\mathcal{E}$ is both a Π-system and a Λ-system, then $\mathcal{E}$ is a σ-algebra.*

Lemma 2. *Let E be a set and $\Pi \subseteq P(E)$ be a Π-system on E. Then $\Lambda(\Pi)$ is a σ-algebra.*

For the proof, use the above properties of the Π-system and a Λ-system along with the properties of measures to check for the requirements to be a sigma-algebra.

1.4 Measurable Functions

Measurable Map: Let $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ be measurable spaces and $f : E \to E'$ be a function. We say f is measurable if $\forall A \in \mathcal{E}' \implies f^{-1}(A) \in \mathcal{E}$.

Random Variable: If $(E, \mathcal{E})$ is a probability space, then f is called a random variable.

Simple Function: A measurable function $s : E \to [0, \infty]$ is called a *simple function* if it can be written as

$$s(x) = \sum_{k=0}^{n} a_k \chi_{A_k}(x),$$

where $a_k \in [0, \infty]$ and $A_k \in \mathcal{E}$ and are disjoint, and χ_{A_k} is the characteristic function defined by

$$\chi_{A_k}(x) = \begin{cases} 1 & \text{if } x \in A_k, \\ 0 & \text{if } x \notin A_k. \end{cases}$$

We denote the set of simple functions by S_+.

1.5 Lebesgue Integration

Simple functions: Continuing the previous notations for simple functions, the integral for s over set $A \in \mathcal{E}$ is defined as

$$\int_A s d\mu = \sum_{k=0}^{n} a_k \mu(A \cap A_k)$$

For a general measurable function f, Lebesgue integration over a set $A \in \mathcal{E}$ is defined as follows:

$$\int_A f d\mu = \sup_{s \in S_+, s \leq f} \int_A s d\mu$$

Some properties:

- $f \leq g \implies \int_A f d\mu \leq \int_A g d\mu$
- $\mu(A) = 0 \implies \int_A f d\mu = 0$
- $\int_E \chi_A f d\mu = \int_A f d\mu$.

2 Three Big Convergence Results

2.1 Monotone Convergence Theorem

Let $f : E \to [0, \infty]$ be a measurable function and $(f_n)_{n \geq 0}$ be a sequence of measurable functions such that for all $x \in E$ and for all $n \in \mathbb{N}$, $f_n(x) \leq f_{n+1}(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$. Then, for all $A \in \mathcal{E}$,

$$\int_A f \, d\mu = \int_A \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu.$$

Proof Idea: Let $L = \lim_{n\to\infty} \int_A f_n \, d\mu$. From the definition of the integral and the monotonicity of the integrals, $L \leq \int_A f d\mu$.

To show the other side, use simple functions. Take a simple function $s \leq f$ and a fixed a such that $0 < a < 1$. For each n, define $A_n = \{x | f_n(x) \geq as(x)\}$. Since $f_{n+1} \geq f_n$, A_n is an increasing family. Furthermore, $f \geq s$ by definition and $\lim_{n\to\infty} f_n = f$. Thus,

$$\bigcup_{n\geq 0} A_n = E$$

We integrate f_n on A_ns. We know that $\int_{A_n} f_n d\mu \geq a \int_{A_n} s d\mu$. Taking $n \to \infty$,

$$\int_E f_n d\mu \geq a \sup_{s\leq f} \int_E s d\mu = a \int_E f d\mu$$

2.2 Fatou's Lemma

Let $(f_n)_{n\geq 0}$ be a sequence of non-negative measurable functions on a measure space $(E, \mathcal{E}, \mu)$. Then,

$$\int_E \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu.$$

Proof Idea: Use monotone convergence theorem on $g_n = \inf_{m\geq n} f_n$. The inequality comes from the monotonicity of integrals. We need MCT to ensure that the RHS actually has a limit.

2.3 Dominated Convergence Theorem

Let $(E, \mathcal{E}, \mu)$ be a measure space and $f_n : E \to \mathbb{R}$ be a sequence of measurable functions that converge pointwise to a function f on μ-almost everywhere. Suppose there exists a $g \in L^1(E, \mathcal{E}, \mu)$ such that for μ-almost everywhere, $|f(x)| \leq g(x)$ and $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in E$. Then,

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof Idea: Define $h_n = (2g - |f - f_n|)$, which is non-negative due to triangle inequality. Then apply Fatou's lemma.

3 Product Measures

Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measure spaces. The product σ-algebra is defined by

$$\mathcal{E}_1 \otimes \mathcal{E}_2 = \sigma(\{A \times B | A \in \mathcal{E}_1, B \in \mathcal{E}_2\})$$

3.1 Fubini's Theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, and let $f : X \times Y \to \mathbb{R}$ be a measurable function. If f is integrable with respect to the product measure $\mu \times \nu$, then

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y).$$

3.2 Tonelli's Theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, and let $f : X \times Y \to [0, \infty]$ be a non-negative measurable function. Then

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y).$$

3.3 General Change of Variables Formula

Let U and V be open subsets of $\mathbb{R}^d$ and $\Phi : U \to V$ be a C^1 bijection such that the Jacobean $J\Phi(x) \neq 0 \forall x \in U$. Then if $f : U \to [0, \infty]$ is measurable, then

$$\int_U f(\Phi) d\lambda_{\mathbb{R}^d} = \int_V f(y) J\Phi^{-1}(y) d\lambda_{\mathbb{R}^d}$$

4 Inequalities

4.1 Markov's Inequality

Let $g : E \to [0, \infty]$ be integrable. Then for $\lambda > 0$, we have

$$\mu(\{g \leq \lambda\}) \leq \frac{1}{\lambda} \int_E g \cdot d\mu$$

4.2 Hölder's Inequality

Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(E, \mathcal{E}, \mu)$ be a measure space and $f, g : E \to [0, \infty]$ be measurable functions. Then,

$$\int_E f(x) g(x) \, d\mu(x) \leq \left(\int_E f(x)^p \, d\mu(x) \right)^{\frac{1}{p}} \left(\int_E g(x)^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

Equality holds if and only if f^p and g^q are proportional, i.e., $f^p = \lambda g^q$ for some $\lambda \geq 0$.

4.3 Jensen's Inequality

Let $(E, \mathcal{E}, \mu)$ be a measure space, $f : E \to \mathbb{R}$ be a measurable function, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, for any $f \in L^1(E, \mathcal{E}, \mu)$,

$$\varphi \left(\int_E f \, d\mu \right) \leq \int_E \varphi(f) \, d\mu.$$

Equality holds if and only if f is almost surely constant or φ is affine.

5 Different Notions of Convergence

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